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Counting States of Black Strings with Traveling Waves II

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Abstract

We extend our recent analysis of the entropy of extremal black strings with traveling waves. We previously considered waves carrying linear momentum on black strings in six dimensions. Here we study waves carrying angular momentum on these strings, and also waves carrying linear momentum on black strings in five dimensions. In both cases, we show that the horizon remains homogeneous and compute its area. We also count the number of BPS states at weak string coupling with the same distribution of linear and angular momentum, and find complete agreement with the black string entropy.

I. INTRODUCTION

In a recent paper [1], we studied a class of supersymmetric solutions to string theory which contain regular event horizons and depend on arbitrary functions. These solutions describe extremal black strings with traveling waves and have an inhomogeneous distribution of momentum along the string. (Solutions of this type were first discussed in [2].) It was shown that for each traveling wave, the Bekenstein-Hawking entropy agreed precisely with the number of BPS states at weak string coupling having the same momentum distribution as the black string. This extended recent work [3 - 11] in which the microstates corresponding to black hole entropy were identified. These earlier investigations reproduced the gravitational entropy of certain black holes (or translationally invariant black strings) by counting the number of bound states of D-branes [12,13] with fixed total momentum. We found that this agreement extends to the case where an (essentially) arbitrary momentum distribution is fixed and the corresponding black string is not translationally invariant.

In [1], we considered six dimensional black strings with a constant internal four dimensional space (so that the total spacetime has ten dimensions). We studied two different types of traveling waves; one carrying momentum along the string, and the other carrying momentum both along the string and in the internal four dimensional space. Here, we extend this work in two directions. In section II we add waves carrying angular momentum to the six dimensional black string. We compute the horizon area and compare the Bekenstein-Hawking entropy with the number of D-brane states at weak coupling with a given angular momentum distribution. Once again we find complete agreement. In section III, we consider five dimensional black strings (which yield four dimensional black holes upon dimensional reduction). To generalize the treatment, the size of the internal five torus is allowed to vary in spacetime. We study waves carrying linear[1] momentum in all spacelike directions and examine the solutions near the horizon. We then compare the horizon area and the number of D-brane states with the given momentum distributions. As before, the Bekenstein-Hawking entropy of the black string agrees with the counting of states.

It was also shown in [1] that the traveling waves do not affect the local geometry of the event horizon; it remains a homogeneous surface. Physically, this is because the waves become purely outgoing near the horizon. We will see that the same remains true for all the waves studied here.

II. ANGULAR MOMENTUM WAVES FOR 6D BLACK STRINGS

As in [1], we consider six dimensional black string solutions to type IIB string theory compactified on a four torus with volume V. We assume one additional spatial direction is compactified to form a circle of length L and choose $L \gg \mathsf{V}^{1/4}$ so that the solutions

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[1] Rotating black holes in four dimensions are not supersymmetric and five dimensional black strings do not support traveling waves with angular momentum.

resemble strings in six dimensions. We are interested in solutions with nonzero electric and magnetic charges associated with the RR three-form; in the limit of weak string coupling, these charges are carried by D-onebranes and D-fivebranes.

Solutions with these charges have a regular event horizon even in the extremal limit. Furthermore, such extremal black strings have a null Killing field $\partial/\partial v$ so that one can use the observations of [14,15] to add traveling waves. Several types of waves carrying linear momentum were considered in [1]. We now study a different class of waves traveling along the string; these waves will carry angular momentum.

A. Traveling Waves Carrying Angular Momentum

In [1], we investigated the metric

$$ds^2 = \left(1+\frac{r_0^2}{r^2}\right)^{-1}\left[-dudv + \left(\frac{p(u)}{r^2} - 2\ddot{f}_i(u)y^i - 2\ddot{h}_i(u)x^i\right)du^2\right] + \left(1+\frac{r_0^2}{r^2}\right)dx_idx^i + dy_idy^i \tag{2.1}$$

which describes a black string carrying a 'longitudinal wave' $p(u)$, an 'internal wave' $f_i(u)$, and an 'external wave' $h_i(u)$. Each of these waves carry momentum in a different direction [16,17]. In the metric (2.1), dots denote d/du, the coordinates y^i label points on the T^4, the x^i label points in the four dimensional asymptotically flat space, and $r^2 = x_ix^i$. In addition, $u = t - z$, $v = t + z$ where z is a coordinate on the S^1. For this solution, the dilaton is constant and the (integer normalized) charges associated with the RR three-form take the values

$$Q_1 = \frac{Vr_0^2}{g}, \qquad Q_5 = \frac{r_0^2}{g} \tag{2.2}$$

where g denotes the string coupling.

We may now add angular momentum to this black string as described in [18]. To preserve supersymmetry, one needs equal amounts of angular momentum in two orthogonal planes [6]. Although [18] considered only uniformly rotating strings, the angular momentum density may again be taken to be a function of u without otherwise changing the metric or matter fields [19]. This is directly analogous to the situation for longitudinal momentum. Writing the metric on the three sphere in the form $d\Omega_3^2 = d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2$, we obtain

$$\begin{aligned} ds^2 = &\left(1+\frac{r_0^2}{r^2}\right)^{-1}\left[-dudv + \frac{p(u)}{r^2}du^2 + \frac{2\gamma(u)}{r^2}(\sin^2\theta d\varphi - \cos^2\theta d\psi)du\right] \\ &+ \left(1+\frac{r_0^2}{r^2}\right)(dr^2 + r^2 d\Omega_3^2) + dy_idy^i, \end{aligned} \tag{2.3}$$

which describes a black string with angular momentum density $\gamma(u)/\kappa^2$. The constant κ^2 is given by

$$\kappa^2 \equiv \frac{4G_{10}}{\pi \mathsf{V}} = \frac{2\pi g^2}{V}, \tag{2.4}$$

where we have used the fact that the ten dimensional Newton's constant is related to the string coupling by $G_{10} = 8\pi^6 g^2$ in units with $\alpha' = 1$ and set $\mathsf{V} = (2\pi)^4 V$. This black string has total angular momentum

$$J_\varphi = -J_\psi = \kappa^{-2} \int_0^L \gamma(u)\ du \tag{2.5}$$

and longitudinal momentum

$$P = \kappa^{-2} \int_0^L p(u)\ du. \tag{2.6}$$

For simplicity, we have set the internal wave $f_i(u)$ and the external wave $h_i(u)$ to zero in (2.3). These waves could have been retained without altering the conclusions below.

The longitudinal wave $p(u)$, on the other hand, cannot be set to zero in the presence of angular momentum. We will see that, in order for the horizon to have a finite area, the longitudinal wave $p(u)$ must be at least $\gamma^2(u)/r_0^4$ at each point along the string. Of course, both $p(u)$ and $\gamma(u)$ must be periodic with period L.

It turns out that, near the horizon, this metric effectively coincides with a metric studied in [1]. As a result, after making the correspondence clear, we may simply read off the desired features of the horizon. This is done by replacing ϕ, ψ with the new coordinates

$$\begin{aligned} \tilde{\varphi} &= \varphi + \frac{\beta(u)}{(r^2 + r_0^2)^2}, \\ \tilde{\psi} &= \psi - \frac{\beta(u)}{(r^2 + r_0^2)^2}, \\ \text{with } \beta(u) &= \int^u \gamma(u')\ du'. \end{aligned} \tag{2.7}$$

The metric then takes the somewhat complicated form

$$\begin{aligned} ds^2 = &\left(1 + \frac{r_0^2}{r^2}\right)^{-1} \left[-du dv + \left(\frac{p(u)}{r^2} - \frac{\gamma^2(u)}{r_0^4 r^2}\right) du^2\right] \\ &+ \left(1 + \frac{r_0^2}{r^2}\right) \left(dr^2 + r^2[d\theta^2 + \sin^2\theta d\tilde{\varphi}^2 + \cos^2\theta d\tilde{\psi}^2]\right) + dy_i dy^i \\ &+ \frac{16 r^2 \beta^2}{(r^2 + r_0^2)^5} dr^2 + \frac{8\beta r}{(r^2 + r_0^2)^2} dr(\sin^2\theta d\tilde{\varphi} - \cos^2\theta d\tilde{\psi}) \\ &+ \frac{r^2\gamma^2}{r_0^4 (r^2 + r_0^2)^3}(r^2 + 2r_0^2) du^2. \end{aligned} \tag{2.8}$$

Note, however, that the first two lines of (2.8) give just the metric (2.1) for a black string with the longitudinal wave $p - \gamma^2/r_0^4$ and no angular momentum. The last two lines may be considered as correction terms; they are all of sub-leading order near the horizon. In fact, the terms on the third line of (2.8) vanish on the horizon and require no further work. Because the horizon lies at $u = \infty$, the du^2 term (on the last line above) does not vanish on the horizon; however, this term turns out to be of the same form as the subleading order terms discussed in the appendix of [1]. As a result, the techniques used there apply to this

case as well and show both that the metric is at least C^0 at the horizon and that the horizon is locally a homogeneous surface. Again, the waves do not affect the local horizon geometry.

To compute the area of the horizon, we must study both its local *and* its global structure. From (2.7), we may deduce the effects of our coordinate transformation on the global identifications. Note that the new angles $(\tilde{\varphi}, \tilde{\psi})$ remain periodic with period 2π on the three-sphere, but that under the identification $z \to z - L$ the new angular coordinates acquire a shift: $(\tilde{\varphi}, \tilde{\psi}) \to (\tilde{\varphi} + (r^2 + r_0^2)^{-2} \int_0^L \gamma du,\ \tilde{\psi} - (r^2 + r_0^2)^{-2} \int_0^L \gamma du)$. This change in global structure is analogous to changing the modular parameter of a torus; the global structure is different, but the volume remains unchanged. In this case, the new identifications imply that a translation along the horizon is accompanied by a rotation of the three-sphere in the two orthogonal planes. It follows that the horizon area is the same as for a longitudinal traveling wave with profile

$$\tilde{p}(u) = p(u) - \frac{\gamma^2(u)}{r_0^4}. \tag{2.9}$$

As explained in [1], to write this area in a simple form we must introduce a function σ, periodic in u, which is related to $\tilde{p}(u)$ by $\sigma^2 + \dot{\sigma} = r_0^{-4}\tilde{p}$. The horizon area is then

$$A = 2\pi^2 r_0^4 \mathsf{V} \int_0^L \sigma(u)\ du. \tag{2.10}$$

When $\tilde{p}$ satisfies the 'slowly varying condition'

$$\tilde{p}^{3/2} \gg r_0^2 |\dot{\tilde{p}}|, \tag{2.11}$$

we have $\sigma = \sqrt{\tilde{p}/r_0^4}$ and the area takes the form $A = 2\pi^2 r_0^2 \mathsf{V} \int_0^L \sqrt{\tilde{p}} du$. The corresponding Bekenstein-Hawking entropy is then

$$S_{BH} = \frac{A}{4G_{10}} = \sqrt{2\pi Q_1 Q_5} \int_0^L \sqrt{\tilde{p}(u)/\kappa^2}\ du. \tag{2.12}$$

In the special case where p and γ are constant, the total longitudinal momentum is $P = pL/\kappa^2$ and the total angular momentum is $J = \gamma L/\kappa^2$. Setting $P = 2\pi N/L$, we obtain $S_{BH} = 2\pi\sqrt{Q_1 Q_5 N - J^2}$ as expected [6,18].

B. Counting BPS States

We now show that the exponential of the entropy (2.12) yields the number of BPS states at weak string coupling with the same distribution of angular momentum and longitudinal momentum as the black string. Recall that at weak coupling the black string corresponds to a collection of D-fivebranes and D-onebranes. The low energy excitations are described by a supersymmetric sigma model in $1+1$ dimensions (where the spatial direction corresponds to our 'string' direction z) containing $4Q_1Q_5$ bosonic fields and an equal number of fermionic fields.

It was shown in [6] that the number of D-brane configurations with total momentum P and angular momentum J agrees with the Bekenstein-Hawking entropy of an extreme five dimensional black hole, which is equivalent to a homogeneous black string, i.e. (2.3) with $p = P\kappa^2/L,\ \gamma = J\kappa^2/L$. Our strategy will be to consider the string to be made up of many short homogeneous segments and then to apply the results of [6] to each one in turn. The key point is thus to show that the fields on different segments may be treated independently. The argument is identical to the one given in [1], so we will only sketch the derivation below.

Recall that one cannot fix the *exact* value $j(u)$ of a current in a 1+1 quantum field theory. The reason is simply that $j(u)$ and $j(u')$ do not commute; for example, when j is the momentum density, its Fourier modes satisfy the Virasoro algebra. We therefore take a 'mesoscopic' viewpoint for our discussion. That is, we imagine that we use an apparatus which can resolve the system only down to a 'mesoscopic' length scale l which is much larger than the 'microscopic' length scale (discussed below) on which quantum effects are relevant. We will therefore divide the spacetime into L/l intervals Δ_a ($a \in \{1, ..., L/l\}$) of length $l \ll L$. If our instruments find a momentum distribution $p(u)/\kappa^2$ and an angular momentum distribution $\gamma(u)/\kappa^2$, this simply means that the interval Δ_a contains a momentum $P_a = \kappa^{-2}\int_{\Delta_a} p(u)du$ and an angular momentum $J_a = \kappa^{-2}\int_{\Delta_a}\gamma(u)du$; we cannot resolve p and γ on smaller scales. Of course, it would be meaningless for us to assign a distribution (p,γ) which has structure on scales of size l or smaller. As a result, l should be much smaller than $p/|\dot{p}|$ and $\gamma/|\dot{\gamma}|$ (and therefore $\tilde{p}/|\dot{\tilde{p}}|$), the 'macroscopic' length scales set by the variation of the wave profile (p,γ).

We shall take the idea that l is much larger than any microscopic length scale to mean that the 'level numbers' $P_a l$ and $(P_a l - J_a^2\kappa^2 r_0^{-4})$ are both large ($\gg Q_1Q_5$). These conditions imply that the wavelength of a typical excited mode with momentum $\tilde{p}l/\kappa^2$ is much less than l, and they are just the conditions imposed in [6] to enable a counting of states on a string of length l. Thus we must choose l to satisfy $\tilde{p}/\dot{\tilde{p}} \gg l \gg \sqrt{Q_1Q_5\kappa^2/\tilde{p}}$. Such an l can exist only when

$$\tilde{p}^{3/2} \gg |\dot{\tilde{p}}|\sqrt{Q_1Q_5\kappa^2} = \sqrt{2\pi}r_0^2|\dot{\tilde{p}}|, \tag{2.13}$$

which is equivalent to the slowly varying condition (2.11).

Under this condition, the arguments of [1] show that the entropy is carried by modes of sufficiently high frequency that each interval of length l may be treated as a separate sigma model. The counting of BPS states then reduces to considering states of longitudinal momentum $(P_a - \frac{J_a^2\kappa^2}{lr_0^4})$ and *no* angular momentum; that is, the angular momentum J affects the entropy only by reducing the momentum to be distributed among the entropy carrying modes. Each segment then carries an entropy of $S_a = \sqrt{2\pi Q_1Q_5}\sqrt{P_a l - \frac{J_a^2\kappa^2}{r_0^4}}$ and the total entropy is

$$S = \sqrt{2\pi Q_1Q_5}\int_0^L \sqrt{\tilde{p}/\kappa^2}\, du. \tag{2.14}$$

Thus the number of BPS states agrees with the Bekenstein-Hawking entropy (2.12) of a black string with the corresponding distribution of momentum and angular momentum.

III. TRAVELING WAVES FOR 5D BLACK STRINGS

We now turn to solutions of type IIA string theory which describe black strings in five dimensions. These solutions are related to *four* dimensional black holes. In close analogy with the 6D black strings, one can add traveling waves to the extremal 5D black strings. We will show that the Bekenstein-Hawking entropy of these solutions again corresponds to the number of BPS states at weak string coupling with the same distribution of energy and momentum. This extends the results of [7,8] where this correspondence was shown for the extremal black strings without traveling waves. In addition, we expand the discussion to allow certain moduli of the internal torus to vary across the spacetime.

We will consider solutions to type IIA string theory carrying magnetic charge with respect to the RR two-form, electric charge with respect to the RR four-form, and magnetic charge with respect to the NS-NS three-form. At weak coupling, these charges are carried by a D-sixbrane, D-twobrane, and solitonic fivebrane, respectively[2]. We take six dimensions to be compactified to form a torus and assume translational symmetry in five of these dimensions. The sixth (z) direction will have a length L much longer than the others, and will be the direction in which the waves propagate. Hence these solutions describe black strings with traveling waves in five dimensions. Four of the toroidal directions (labeled by $y^a,\ a = 1,2,3,4$) form a torus of volume $\mathsf{V} = (2\pi)^4 V$, and one (w) forms a circle of length $\tilde{L}$. The other four dimensions $(t, x^i,\ i = 1,2,3)$ will be asymptotically flat.

A. Classical Black String Solutions

The solution of type IIA string theory describing five dimensional black strings with the above charges was found in [22]. It is characterized by three harmonic functions

$$H_2(r) = 1 + \frac{r_2}{r}, \qquad H_5(r) = 1 + \frac{r_5}{r}, \qquad H_6(r) = 1 + \frac{r_6}{r}, \tag{3.1}$$

where $r^2 = x_i x^i$. The constants $r_2,\ r_5,\ r_6$ are related to the integer charges by

$$Q_2 = \frac{2r_2 V}{g}, \qquad Q_5 = \frac{r_5 \tilde{L}}{\pi}, \qquad Q_6 = \frac{2r_6}{g}. \tag{3.2}$$

Introducing the null coordinates $u = t - z$ and $v = t + z$ and using the flat metric δ_{ij} and δ_{ab} to raise and lower the indices $i \in \{1,2,3\}$ and $a \in \{1,2,3,4\}$, the Einstein metric for the black string takes the form

$$\begin{aligned} ds^2 = H_2^{3/8} H_5^{6/8} H_6^{7/8} \Big[& - H_2^{-1} H_5^{-1} H_6^{-1} du dv \\ & + H_5^{-1} H_6^{-1} dy_a dy^a + H_2^{-1} H_6^{-1} dw^2 + dx_i dx^i \Big] \end{aligned} \tag{3.3}$$

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[2] One can also form five dimensional black strings or four dimensional black holes with other choices of charges [8,20,21]. This choice was first used in [7]. We follow the conventions of [10].

and the dilaton is $e^{2\phi} = H_2^{1/2} H_5 H_6^{-3/2}$. The horizon is at $r = 0$, $u = \infty$. Note that ϕ approaches a constant both at infinity and at the horizon. When all three harmonic functions are equal, $H_2 = H_5 = H_6 \equiv H$, the dilaton vanishes and the metric reduces to

$$ds^2 = -H^{-1} du dv + dy_a dy^a + dw^2 + H^2 dx_i dx^i \tag{3.4}$$

which is just the product of a five torus and the extremal black string solution of the five dimensional Einstein-Maxwell theory [23].

Since the solutions (3.3) all possess the null Killing vector field $\partial/\partial v$, we may again add traveling waves using the methods of [14,15]. The result is a metric of the form

$$\begin{aligned} ds^2 = H_2^{3/8} H_5^{6/8} H_6^{7/8} \Big[& H_2^{-1} H_5^{-1} H_6^{-1} du \, [-dv + K(u,x,y) du] \\ & + H_5^{-1} H_6^{-1} dy_a dy^a + H_2^{-1} H_6^{-1} dw^2 + dx_i dx^i \Big] \end{aligned} \tag{3.5}$$

where K satisfies

$$(\partial_i \partial^i + \partial_a \partial^a + \partial_w \partial_w) K = 0. \tag{3.6}$$

That is, K is harmonic in the toroidal (w, y^a) and asymptotically flat (x^i) coordinates, but has arbitrary u dependence. As a result, K contains free functions that describe traveling waves along the 'string direction' (z). Since the surface $r = 0$ is a coordinate singularity in (3.3), we only require (3.6) to hold for $r \neq 0$. Nonetheless, the horizon will be a regular surface for all of the metrics we consider. For the moment, we will ignore the details of the compactifications; they will be discussed below.

We wish to consider only waves that are in some way 'anchored' to the black string, i.e., waves that either become pure gauge or have unphysical singularities when the black string is removed. Such waves were first discussed in [16,17] in connection with a fundamental string and later in [1] for a six dimensional black string. In the present context, the waves are given by

$$K = \frac{p(u)}{r} - 2\ddot{f}_a(u) y^a - 2\ddot{b}(u) w - 2\ddot{h}_i(u) x^i. \tag{3.7}$$

As before, the p term represents 'longitudinal waves,' the f_a and b terms represent 'internal waves', and the h_i term represents 'external waves.' We will see that this terminology corresponds to the various directions in which the waves carry momentum. Since the internal and external waves are clearly negligible compared to the longitudinal wave near the horizon $(r = 0)$, one might expect that they do not contribute to the horizon area. We will see that this is indeed the case.

With the waves (3.7), the metric (3.5) is neither asymptotically flat nor translationally invariant in the toroidal directions. Both difficulties can be resolved by transforming to coordinates (u, v', w', x', y') which are related to those above through

$$\begin{aligned} v' &= v + 2\dot{f}_a y^a + 2\dot{b} w + 2\dot{h}_i x^i + \int^u (\dot{f}^2 + \dot{b}^2 + \dot{h}^2) du, \\ w' &= w + b, \\ x'^i &= x^i + h^i, \end{aligned}$$

$$y'^a = y^a + f^a, \tag{3.8}$$

where $\dot{f}^2 = \dot{f}_a \dot{f}^a$, $\dot{h}^2 = \dot{h}_i \dot{h}^i$. The metric then takes the form

$$\begin{aligned} ds^2 = H_2^{3/8} H_5^{6/8} H_6^{7/8} \Bigg[& H_2^{-1} H_5^{-1} H_6^{-1} du \Bigg(-dv' + \Big[\frac{p + r_2 \dot{f}^2 + r_5 \dot{b}^2}{r} + (H_2 H_5 H_6 - 1)\dot{h}^2 \Big] du \\ & - \frac{2r_2}{r} \dot{f}_a dy'^a - \frac{2r_5}{r} \dot{b} dw' - 2(H_2 H_5 H_6 - 1)\dot{h}_i dx'^i \Bigg) \\ & + H_5^{-1} H_6^{-1} dy'_a dy'^a + H_2^{-1} H_6^{-1} dw' dw' + dx'_i dx'^i \Bigg]. \end{aligned} \tag{3.9}$$

It is in terms of these coordinates that we make the periodic identifications which compactify the spacetime. Setting $z' = (v' - u)/2$, the large S^1 is defined by the identification $z' \rightarrow z' - L$, or $(u, v', w', x', y') \rightarrow (u + L, v' - L, w', x', y')$, while the small five-torus is defined by the identifications $w' \rightarrow w' + \tilde{L}$, and $y' \rightarrow y' + a_I$ for an appropriate set of four vectors a_I. Clearly, $p(u)$, $\dot{f}(u)$, $\dot{b}(u)$, and $\dot{h}(u)$ must be periodic in u. In addition, we will take $f(u)$, $b(u)$, and $h(u)$ to be periodic themselves. In the ten-dimensional space before compactification, this amounts to considering only black strings with no net momentum in the w', x', and y' directions.

The asymptotic charges can be read directly from the metric (3.9). As in [1], the momentum is not distributed uniformly along the string, and we will match this momentum profile to a configuration of D-branes at weak coupling. Defining[3]

$$\kappa^2 \equiv \frac{4G_{10}}{\mathsf{V}\tilde{L}} = \frac{2\pi^2 g^2}{V\tilde{L}}, \tag{3.10}$$

the black string with traveling waves (3.9) has ADM momentum

$$\begin{aligned} P_{z'} &= \kappa^{-2} \int_0^L du \left[p + r_2 \dot{f}^2 + r_5 \dot{b}^2 + (r_2 + r_5 + r_6)\dot{h}^2 \right] \\ P_a &= \kappa^{-2} r_2 \int_0^L du\ \dot{f}_a \\ P_{w'} &= \kappa^{-2} r_5 \int_0^L du\ \dot{b} \\ P_i &= \kappa^{-2} (r_2 + r_5 + r_6) \int_0^L du\ \dot{h}_i\ , \end{aligned} \tag{3.11}$$

and ADM energy

$$E = \kappa^{-2}(r_2 + r_5 + r_6)L + P_{z'}\ . \tag{3.12}$$

[3] Although κ^2 will play the same role in this section as it did in the previous one, its precise definition is different. In section II, κ^2 was related to the six dimensional Newton's constant; here it is related to Newton's constant in five dimensions.

It is clear from (3.11) that the black string is composed of three different constituents: Oscillations of the same amplitude in different directions result in different amounts of momentum. This is consistent with the weak coupling description in terms of branes wrapped around different directions, but is *not* one would expect from, say, a large rubber band. The spacetime metric thus records the fact that any 'source' must have several components.

B. The Event Horizon

We would like to show that $r = 0$, $u = \infty$ is a horizon in the spacetime (3.5) and to compute the horizon area. The calculations are structurally identical to those performed in [1] but are slightly more complicated due to the presence of the different harmonic functions H_2, H_5, and H_6. We will not present the full details here, but the reader may reconstruct them by copying the steps described in the appendix of [1]. Such techniques suffice to show that the horizon is at least C^0 and its area may be computed using only the leading order behavior of the metric (3.5) near $r = 0$. Setting $r = 4r_2r_5r_6/R^2$, this leading order metric takes the form

$$ds^2 = (r_2^3 r_5^6 r_6^7)^{1/8}\left[4\left(R^{-2}[-dudv + dR^2] + \frac{p(u)}{4r_2r_5r_6}du^2\right) + d\Omega_2^2 + \frac{dy_a dy^a}{r_5r_6} + \frac{dw^2}{r_2r_6}\right]. \quad (3.13)$$

Using the results in [1] one finds that this spacetime has a homogeneous horizon with area

$$A = 8\pi r_2r_5r_6\tilde{L}\mathsf{V}\int_0^L \sigma(u)\, du \quad (3.14)$$

where σ is a periodic function of u satisfying $\sigma^2 + \dot{\sigma} = p/(4r_2r_5r_6)$. In analogy with the six dimensional case, when

$$p^3 \gg (r_2r_5r_6)\dot{p}^2, \quad (3.15)$$

the area becomes $A = 4\pi\sqrt{r_2r_5r_6}\tilde{L}\mathsf{V}\int_0^L \sqrt{p(u)}\, du$ and the Bekenstein-Hawking entropy may be written

$$S_{BH} = \sqrt{2\pi Q_2Q_5Q_6}\int_0^L \sqrt{p(u)/\kappa^2}\, du. \quad (3.16)$$

C. Counting BPS States

Consider the weak coupling limit of the IIA string theory with six dimensions compactified to form a torus as above; one circle has length L much larger than the rest, another has length $\tilde{L}$ and the remaining four have volume $\mathsf{V} = (2\pi)^4 V$. The black strings (3.3) correspond to Q_2 D-twobranes wrapped around L and $\tilde{L}$, Q_5 solitonic fivebranes wrapped around L and V, and Q_6 D-sixbranes wrapped around the entire six torus. Each brane contributes an effective string tension in the L direction given by

$$T_2 = \frac{\tilde{L}}{4\pi^2 g}, \qquad T_5 = \frac{V}{2\pi g^2}, \qquad T_6 = \frac{V\tilde{L}}{4\pi^2 g}. \tag{3.17}$$

It then follows from (3.2) and (3.10) that

$$\frac{r_2}{\kappa^2} = Q_2 T_2, \qquad \frac{r_5}{\kappa^2} = Q_5 T_5, \qquad \frac{r_6}{\kappa^2} = Q_6 T_6. \tag{3.18}$$

We begin by setting $b = h_i = 0$, and proceed as in section III of [1]. We again adopt a mesoscopic viewpoint and divide the string into a number of small intervals. We will consider only the case $Q_6 = 1$, in which the degrees of freedom that contribute to the entropy correspond to oscillations of the twobranes in the sixbranes. There are actually $4Q_2Q_5$ (bosonic) degrees of freedom of this type since the fivebranes 'cut' each twobrane into Q_5 different pieces [7], each with an effective average tension of T_2/Q_5 in the string direction. We interpret the *field* momentum of these bosonic fields as carrying *spacetime* momentum in the internal y^a directions since both generate translations of the twobranes. Recall that a field χ with tension T has momentum density $T\dot{\chi}$, and the black string has momentum density $r_2\dot{f}_a/\kappa^2$ in the y^a direction (3.11). Thus the condition for the Q_2Q_5 fields χ_A associated with fluctuations in the y_1 direction to have the same momentum as the black string is

$$\frac{T_2}{Q_5}\sum_{A=1}^{Q_2Q_5} \dot{\chi}_A = \frac{r_2\dot{f}_1}{\kappa^2} = Q_2 T_2 \dot{f}_1. \tag{3.19}$$

It follows that f_1 is just the average fluctuation of the D-branes. We impose similar conditions for the other three internal directions (y_2, y_3, y_4) transverse to the twobranes.

The counting may now be performed just as in [1]. Applying our 'mesoscopic' viewpoint, we divide the string into segments of length l and restrict only the average values of the oscillations χ_A to satisfy (3.19) on each segment. Thus the internal momentum is carried by the zero mode in each segment. An internal momentum distribution $Q_2T_2\dot{f}^a$ among Q_2Q_5 twobranes of tension T_2/Q_5 requires a longitudinal momentum of at least $Q_2T_2\dot{f}^2$. The remaining longitudinal momentum is just p/κ^2, and the entropy arises from distributing this momentum arbitrarily. When each interval is highly excited, this gives the result

$$S = \sqrt{2\pi Q_2 Q_5}\int_0^L \sqrt{p(u)/\kappa^2}\, du. \tag{3.20}$$

As usual, the intervals can be highly excited only when $p^3 \gg (r_2r_5r_6)\dot{p}^2$ so that the slowly varying condition (3.15) holds. Since $Q_6 = 1$, (3.20) agrees with the Bekenstein-Hawking entropy (3.16) of the black string.

Now consider the waves $b(u)$ and $h_i(u)$. In general, the oscillation χ of a string with tension T has momentum $T\dot{\chi}$ in the transverse direction and $T\dot{\chi}^2$ in the longitudinal direction. It thus follows from (3.11) and (3.18) that the wave $b(u)$ is naturally interpreted as corresponding to macroscopic oscillations of the fivebranes, and the wave $h_i(u)$ is interpreted as coordinated oscillations of all the branes together in the macroscopic directions. The contributions of $b(u)$ and $h_i(u)$ to both the longitudinal and transverse momenta can be accounted for in this way. At weak string coupling, there are a small number of fields which

describe these oscillations. They act just like the fields χ described above: By requiring that their field momenta agree with the transverse momentum of the black string, we 'use up' part of the longitudinal momentum. The remaining longitudinal momentum is just $p(u)/\kappa^2$, independent of $b(u)$ and $h_i(u)$. Since the number of such fields is small compared to Q_2Q_5, they do not change the number of degrees of freedom which can carry the longitudinal momentum to leading order. Thus the entropy is again given by (3.20) in agreement with the Bekenstein-Hawking entropy of the black string.

IV. DISCUSSION

We have considered two new families of solutions containing black strings with traveling waves. One describes six dimensional black strings with waves carrying angular momentum, and the other describes five dimensional black strings with waves carrying linear momentum in various directions. We computed the horizon area, and counted the number of BPS states at weak string coupling with the same distribution of momentum and angular momentum. In each case, the Bekenstein-Hawking entropy agreed with the number of microscopic string states. Combined with the results of [1], this leads to the following conclusion: *In every case where the entropy of an extreme black hole has been understood in terms of string states (five dimensional black holes including rotation, and four dimensional black holes without rotation) one can apply the same counting arguments in a quasilocal way along the D-brane, to explain the entropy of an inhomogeneous black string.*

Note, however, that this quasilocal picture is lost near the horizon of the black string. Even though the horizon always adjusts itself so that its area agrees with the counting of states, it does so globally - not locally. Locally, the horizon remains homogeneous, and is largely unaffected by the traveling waves. This is similar to what happens with other disturbances in the spacetime. For example, one can add another sixbrane at a point x_0 outside the black string by simply replacing H_6 in (3.3) with

$$H_6 = 1 + \frac{r_6}{r} + \frac{g}{2|x - x_0|} \tag{4.1}$$

It is clear that the effect of this new sixbrane becomes negligible near the horizon $r = 0$; both the area and local geometry of the horizon remains unchanged. This is expected since the area of the horizon is a measure of the number of internal states, and it should not be affected by the presence of matter (such as the new sixbrane) with which the string does not interact.

Let us now return to the six dimensional black string with angular momentum. Recall that in section II we imparted angular momentum to our string by adding the wave $\gamma(u)$, which left the linear momentum density unchanged. However, as described in [16,17], one may also give the black string angular momentum by adding appropriate external waves $h_i(u)$. Roughly speaking, the wave $\gamma(u)$ corresponds to spinning the string, while the external wave $h_i(u)$ corresponds to a string gyrating so as to act like a rotating helical coil. If we specify both the angular and the linear momentum density of the string, the contribution to the angular momentum from the 'spin waves' $\gamma(u)$ and the 'gyrating external waves'

$h_i(u)$ are uniquely determined. We have seen that the Bekenstein-Hawking entropy of the resulting spacetime agrees with the counting of weak coupling bound states whenever our mesoscopic picture is well-defined, for any combination of gyration and rotation.

In addition, it is illustrative to consider a different point of view. Suppose one wanted to study rotating black *holes*. Both spinning and gyrating black strings reduce to rotating black holes in five dimensions, at least after appropriate averaging (see [16,17]). However, when considering the reduced solution it does not seem appropriate to specify the *distribution* of momentum along the string. As a result, we wish to consider the collection of D-brane states corresponding to fixed total momentum P and angular momentum J, but without any further restriction. Such states will in general include both spin waves and gyrating waves; we would like to know which contribution dominates (if any) and to see that the result remains compatible with the entropy of the five dimensional black hole.

Suppose that we consider the somewhat smaller collection of states for which the spin waves contribute an angular momentum J_{spin} and the gyrating waves contribute an angular momentum J_{gyro}, where $J = J_{spin} + J_{gyro}$. The relative contributions of such states can be determined from the corresponding entropy. Recall that both waves effect the entropy by reducing the longitudinal momentum that may be freely distributed among the various modes. From section II, the spin angular momentum reduces the available longitudinal momentum by $J^2_{spin}\kappa^2/Lr_0^4$. On the other hand, the gyrating wave reduces the longitudinal momentum by term of the form $J^2_{gyro}\kappa^2/Lr_0^2A^2$ which depends on the amplitude A of the wave. This is to be expected, as an angular momentum J typically contributes an energy of the form J^2/I where I is a moment of inertia (Lr_0^2/κ^2 is the mass of the string). The division of the angular momentum into spin and gyration is determined by maximizing the entropy (i.e. the available longitudinal momentum) subject to the constraint $J = J_{spin} + J_{gyro}$.

Clearly, the result depends on the allowed radius A of the gyrations. Let us suppose that the gyrations can be arbitrarily large. In this case, the associated black string is quite far from being translationally invariant and appears to be shaking wildly. The averaging process that gives the five dimensional spacetime must then become ill-defined. It seems unlikely than any observer could be 'effectively five-dimensional' in such a setting; any observer would find large discrepancies from the predictions of the five dimensional black hole solution. As a result, such states do not correspond to our picture of a five dimensional black hole. The setting of the problem thus restricts the amplitude A of the gyrations to be much less than r_0, the radius of the five dimensional black hole. With the condition $A \ll r_0$, the entropy is maximized for $J_{spin} = J$, $J_{gyro} = 0$. This value is then overwhelmingly likely and the entropy of the entire collection of states with $J_{spin} + J_{gyro} = J$ and $A \ll r_0$ is

$$S = \sqrt{2\pi Q_1 Q_5}\sqrt{LP - J^2\kappa^2/r_0^4} = 2\pi\sqrt{Q_1 Q_5 N - J^2}, \tag{4.2}$$

in agreement with the entropy of the five dimensional black hole [6].

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